Lecture Notes 2-4: The Precise Definition of the Limit

REVIEW: List our present strategies for determining $\lim_{x\to a} f(x)$, if it exists.

- plugging in numbers.
- using the graph
- using limit laws

What are some of the weaknesses in these approaches?

- plugging in #'s only gives an approximation
- limit laws only apply to familiar functions mostly polys + rational functions.

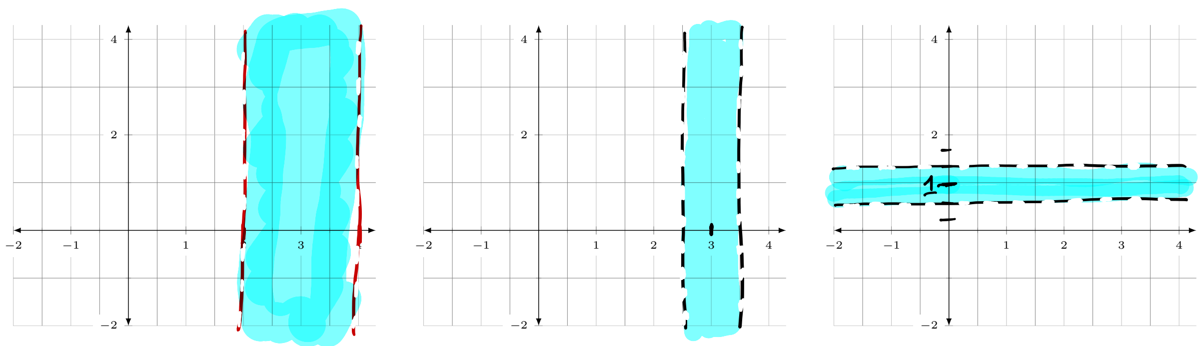
GOALS:

- Experience the precise (or *formal*) definition of the limit.
- Confirm student intuition that there is a lot going on when evaluating limits that may not be crystal clear!)
- "Look under the hood" of the mathematics used in Calculus (& Differential Equations).

PRACTICE PROBLEMS:

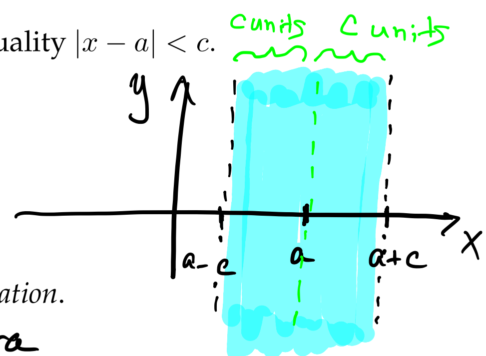
1. Graph the region of the xy-plane satisfying each of the inequalities below.

$|x-3|<1$ $\qquad$ $|x-3|<1/2$ $\qquad$ $|y-1|<1/3$



2. Graph the region of the xy-plane satisfying the inequality $|x-a|<c$.

- In words:
  all points such that their x-coordinate is within c-units of x-value a



3. Re-write the expression $|x-a|<c$ using *interval notation*.

directly from picture

$a-c < x < a+c$

using algebra

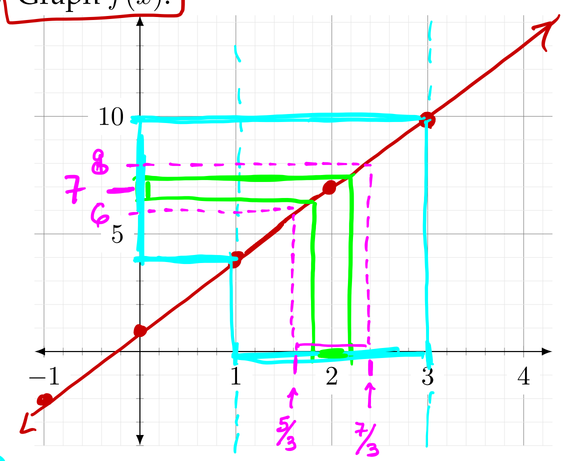
$|x-a|<c$ means $-c < x-a < c$.

So $a-c < x < a+c$

+a to both sides.

4. Let $f(x) = 3x + 1$.

(a) Graph $f(x)$.



(b) If the domain of $f(x)$ is restricted to

$$|x-2| < 1,$$

what would the range of $f(x)$ be? Sketch the intervals representing domain and range on the graph.

picture	algebra
$4 < y < 10$	$1 < x < 3 \Rightarrow 3 < 3x < 9$ (multiply by 3)
	$\Rightarrow 4 < 3x+1 < 10$ (add 1)

(c) If the domain of $f(x)$ is restricted to

$$|x-2| < \frac{1}{5},$$

what would the range of $f(x)$ be? Sketch the intervals representing domain and range on the graph.

$|x-2| < \frac{1}{5}$ means $-\frac{1}{5} < x-2 < \frac{1}{5}$

or $\frac{9}{5} < x < \frac{11}{5}$.

Now $\frac{9}{5} < x < \frac{11}{5}$ implies

multiply by 3: $\frac{27}{5} < 3x < \frac{33}{5}$ which implies

add 1: $\frac{32}{5} < 3x+1 < \frac{38}{5}$

So $\boxed{\frac{32}{5} < y < \frac{38}{5}}$ or $\boxed{|y-7| < \frac{3}{5}}$

(d) **We are now going to switch our point of view so READ CAREFULLY!** Assume you must "hit a target" in the range. Specifically, assume you need the output of the function to lie in the region $|y-7| < 1$, how would you need to restrict your domain? Is your answer UNIQUE? Give you final answer in the form $|x-a| < c$, for some a and c.

$y=6 \Rightarrow x = \frac{5}{3}$, $y=8 \Rightarrow x = \frac{7}{3}$

So we need $\frac{5}{3} < x < \frac{7}{3}$ or $|x-2| < \frac{1}{3}$

unique? No! Any interval within the pink one would hit the target. The green one, for example would work. Note a smaller interval will not COVER completely the target.

(e) Repeat the previous question but replace the target interval $|y-7| < 1$ with $|y-7| < \frac{1}{10}$.

$y = 7 + \frac{1}{10} = \frac{71}{10} = 3x+1$

$\frac{61}{10} = 3x$

$x = \frac{61}{30}$

$y = 7 - \frac{1}{10} = \frac{69}{10} = 3x+1$

$\frac{59}{10} = 3x$

$x = \frac{59}{30}$

So $\frac{59}{30} < x < \frac{61}{30}$

$-\frac{1}{30} < x-2 < \frac{1}{30}$ or

$|x-2| < \frac{1}{30}$

(f) Repeat the previous question but replace the target interval with $|y-7| < \epsilon$, where ϵ is some unknown but fixed positive number. (Note the symbol ϵ is called *epsilon* and in mathematics is very nearly always used to represent some really small positive number, 0.0000001, for example.)

If $y = 7+\varepsilon = 3x+1$, then $x = \frac{6+\varepsilon}{3} = 2 + \frac{\varepsilon}{3}$.

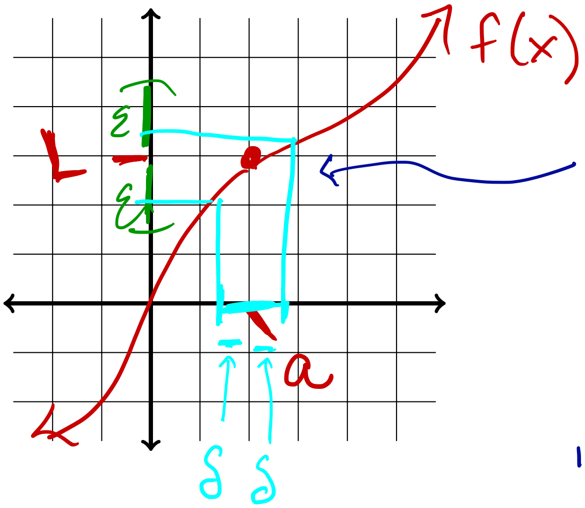
If $y = 7-\varepsilon = 3x+1$, then $x = \frac{6-\varepsilon}{3} = 2 - \frac{\varepsilon}{3}$.

So $2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$ or $|x-2| < \frac{\varepsilon}{3}$

THE PRECISE DEFINITION OF THE LIMIT:

We say $\lim_{x \to a} f(x) = L$, if for *every* number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$



HEY! given a target: $|y - L| < \varepsilon$,

we found an interval for the domain: $|x - a| < \delta$,

So that the function sends our domain into the target. ALWAYS

5. Use the precise definition of the limit to show $\lim_{x \to 2} 3x + 1 = \boxed{7}$.

This WHOLE thing is the answer.

(i) Given $\varepsilon > 0$, pick $\delta = \varepsilon/3$.

(ii) Assume $|x - 2| < \frac{\varepsilon}{3} = \delta$.

(iii) Observe, if $-\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}$

then $2 - \frac{\varepsilon}{3} < x < 2 + \frac{\varepsilon}{3}$

$6 - \varepsilon < 3x < 6 + \varepsilon$

$7 - \varepsilon < 3x + 1 < 7 + \varepsilon$.

So $7 - \varepsilon < y < 7 + \varepsilon$ or $|y - 7| < \varepsilon$

Step (i): Pick δ.

Step (ii): Assume $|x - a| < \delta$

Step (iii): STARTING with your assumption in (ii), proceed with the algebra determined by the given function to get back to y.

6. In words in English, write down how to apply the precise definition of the limit. Assume you are given a function, $f(x)$ and an x-value a and asked to find $\lim_{x \to a} f(x)$.

1. Identify the specific $f(x)$ and a you are given.
2. Figure out what L is.
3. Set $y = L + \varepsilon$ (and $L - \varepsilon$) and solve for x.
4. Part 3. will give an interval for x, Pick δ to fall within it.
5. Now and only Now, follow the exact structure in the blue box above.